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Note

Upper minus domination in a claw-free cubic graph☆

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Abstract

We show in this paper that the upper minus domination number $\Gamma^-(G)$ of a claw-free cubic graph G is at most $\frac{1}{2}|V(G)|$.

Keywords: Cubic graph; Claw-free; Minus domination

1. Introduction

Graphs considered in this paper are finite and simple. A graph is called *cubic* if each of its vertices is of degree 3. A graph that contains no induced subgraph isomorphic to $K_{1,3}$ is said to be *claw-free*. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V$, the open neighborhood of v, denoted by $N(v)$, is defined by

$$N(v) = \{u \in V(G) : uv \in E(G)\},$$

and the closed neighborhood of v, denoted by $N[v]$, is defined by

$$N[v] = N(v) \cup \{v\}.$$

If S is a subset of V, we set

$$N(S) = \bigcup_{v \in S} N(v),$$

$$N[S] = N(S) \cup S$$

and

$$N\{S\} = N(S) \backslash S.$$

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If *f* is a weight function on the vertices of G, then, for every $S \subseteq V(G)$, we write

$$f(S) = \sum_{v \in S} f(v)$$

and

$$f^*(S) = f(N[S]).$$

Especially, we write $f^*(v) = f(N[v])$ for $v \in V(G)$.

A *minus dominating function* [1,4] of a graph G is defined as a function $f : V(G) \to \{-1, 0, 1\}$ such that $f^*(v) \geqslant 1$ for every $v \in V(G)$. A minus dominating function *f* is said to be a *minimal minus dominating function* if each minus dominating function h satisfying $h(v) \leqslant f(v)$ for every $v \in V(G)$ is equal to *f*. The *minus domination number* and the *upper minus domination number* of G are denoted by $\gamma^-(G)$ and $\Gamma^-(G)$, are defined by

$$\gamma^-(G) = \min\{f(V(G)) : f \text{ is a minimal minus dominating function of } G\}$$

and

$$\Gamma^-(G) = \max\{f(V(G)) : f \text{ is a minimal minus dominating function of } G\}.$$

A *signed dominating function* [2,5] of a graph G is defined as a function $f : V(G) \to \{-1, 1\}$ such that $f^*(v) \geqslant 1$ for every $v \in V(G)$. The *signed domination number* $\gamma_s(G)$ and the *upper signed domination number* $\Gamma_s(G)$ of a graph G can be similarly defined.

In [6], the author showed that, for every k-regular graph G of order n, $\gamma_s(G) \geqslant n/(k+1)$ if k is even and $\gamma_s(G) \geqslant 2n/(k+1)$ if k is odd. So, $\Gamma_s(G) \geqslant \gamma_s(G) \geqslant n/2$ for a cubic graph G of order n. In [7] the authors showed that, for every cubic graph G, $\Gamma^-(G) \leqslant \frac{5}{8}n$. Other results and developments on the research for minus domination number of graphs can be found in [1,3,4,8–11].

Authors in [7] also posed the following conjecture.

Conjecture. If G is a cubic graph, then $\Gamma^-(G) \leqslant \Gamma_s(G)$.

By our knowledge, no progress has been made on the above conjecture. Motivated by this conjecture, we establish an upper bound for the upper minus domination number of a claw-free cubic graph. We show in this paper that the upper minus domination number $\Gamma^-(G)$ of a claw-free cubic graph G is at most $\frac{1}{2}|V(G)|$. Consequently, $\Gamma^-(G) \leqslant \Gamma_s(G)$ for a claw-free cubic graph G, that is, the above conjecture holds for claw-free graphs.

2. Main result and proof

The following trivial observation is useful for our proof.

Lemma 2.1. *A minus dominating function f of a graph G is minimal if and only if, for every $v \in V(G)$ with $f(v) \geqslant 0$, there is some $u \in N[v]$ such that $f^*(u) = 1$.*

Suppose that G is a connected claw-free cubic graph. Then every vertex of G lies in a triangle in G. We will use C to denote the vertex set of a triangle C in G. Clearly, for every two distinct triangles C and T in G, either $|C \cap T| = 0$ or $|C \cap T| = 2$.

Let *f* be an arbitrary minimal minus dominating function of G. We only need to prove that $f(V(G)) \leqslant \frac{1}{2}|V(G)|$. When $|V(G)| = 4$, we have $G \cong K_4$, and the result can be easily verified. Hence, we suppose $|V(G)| \geqslant 6$ in the sequel.

For $i = -1, 0, 1$, we write

$$V_i = \{v \in V(G) : f(v) = i\}.$$

A vertex v of G is called an (n_{-1}, n_0, n_1)-vertex if

$$|N[v] \cap V_i| = n_i \quad \text{for } i = -1, 0, 1.$$

For $i=1,2,3,4$, we write

$$F_i=\{v\in V(G): f^*(v)=i\}.$$

Since, for each vertex $v\in V(G)$, there are exactly 4 vertices in $N[v]$, we have

$$4f(V(G))=\sum_{u\in V(G)} f^*(u).$$

Consequently, we have

Lemma 2.2. $f(V(G))=\frac{1}{4}(|F_1|+2|F_2|+3|F_3|+4|F_4|)$.

For a triangle T in G, we say T is an *independent triangle* if, for every triangle $C\neq T$ in G we have $T\cap C=\emptyset$. For a vertex $v\in V(G)$, we say v is a *single-triangle vertex* if v is contained in exactly one triangle in G. If $v\in V(G)$ is a single-triangle vertex, we will use T_v to denote the unique triangle containing v. We say a triangle T in G is *critical* if the f-values of the three vertices of T are $-1, 0, 1$, respectively. The following result is also easily observed.

Lemma 2.3. *If T is a critical triangle in G, then for $v\in T$ and $u\in N(v)\backslash T$ we have*

$$v\in F_1 \quad and \quad u\in V_1.$$

Lemma 2.4. *Let v be a vertex of G with $v\in F_4$. Then v is a single-triangle vertex of G. Furthermore, if u is the unique vertex in $N(v)$ such that $u\notin T_v$, then u is also a single-triangle vertex and T_u is a critical triangle.*

Proof. Let $T=\{v,x,y\}$ be a triangle containing v in G. Let u be the unique vertex in $N[v]\backslash T$. Since $v\in F_4$, we have $u,v,x,y\in V_1$. Note that $f^*(x), f^*(y)\geqslant 2$. But, by Lemma 2.1, one of u,x,y belongs to F_1. Hence, the only possibility is that both v and u are single-triangle vertices and T_u is a critical triangle. The result follows. □

Let v and u be the same as in Lemma 2.4. We call T_u the *critical triangle* for v and write $C_v=T_u$. Set

$$F_1(4)=\bigcup_{v\in F_4} C_v$$

and

$$F_3(4)=F_3\cap N(F_1(4)).$$

By Lemma 2.3, we have $F_1(4)\subseteq F_1$.

Lemma 2.5. $|F_3(4)|+2|F_4|\leqslant|F_1(4)|$.

Proof. Set

$$F_4(1)=\{v\in F_4: C_v\cap C_u=\emptyset \text{ for each } u\in F_4\backslash\{v\}\}$$

and

$$F_4(2)=\{v\in F_4: \text{there is } u\in F_4\backslash\{v\} \text{ such that } |C_v\cap C_u|=2\}.$$

Then the vertices in $F_4(2)$ are pairwise matched under the condition that the critical triangles for the two matched vertices have two common vertices. Suppose further that

$$F_4(1)=\{w_1,w_2,\ldots,w_a\}$$

and

$$F_4(2)=\{u_1,u_2,\ldots,u_b,v_1,v_2,\ldots,v_b\}$$

such that $|C_{u_i}\cap C_{v_i}|=2$, $1\leqslant i\leqslant b$.

Let w_i be an arbitrary vertex in $F_4(1)$. By noting that the f-value of the two vertices in C_{w_i} not adjacent to w_i are -1 and 0, respectively, we have $|N(C_{w_i}) \cap F_3| \leqslant 1$. It follows that:

$$|N(C_{w_i}) \cap F_3| + 2 \leqslant |C_{w_i}|, \quad 1 \leqslant i \leqslant a.$$

Let $u_i, v_i \in F_4(2)$, $1 \leqslant i \leqslant b$. Then $N(C_{u_i} \cup C_{v_i}) \cap F_3 = (C_{u_i} \cup C_{v_i} \cup \{u_i, v_i\}) \cap F_3 = \emptyset$. Hence, we have

$$|N(C_{u_i} \cup C_{v_i}) \cap F_3| + 4 \leqslant |C_{u_i} \cup C_{v_i}|, \quad 1 \leqslant i \leqslant b.$$

By summing up the above two families of inequalities, we obtain $|F_3(4)| + 2|F_4| \leqslant |F_1(4)|$. The result follows. □

For a vertex $v \in F_1$, one can easily see that $|N(v) \cap F_3| \leqslant 2$. We say a single-triangle vertex $v \in F_1$ of G is an *undesirable vertex* for f, if $|N(v) \cap F_3| = 2$ and $N\{T_v\} \cap F_1 = \emptyset$. We say a single-triangle vertex $v \in F_1$ of G is a *desirable vertex* for f, if $|N(v) \cap F_3| = 2$ and $|N\{T_v\} \cap F_1| \geqslant 1$.

Lemma 2.6. *Let $v \in F_1$ be a vertex of G. If $|N(v) \cap F_3| = 2$, then v is a single-triangle vertex, $N(v) \cap F_3 = T_v \backslash \{v\}$ and v is either an undesirable or a desirable vertex for f. Furthermore*:

(a) *if v is an undesirable vertex for f, then, $f(v) = 1$ and there is a vertex $x \in T_v$ such that $f(x) = 0$;*
(b) *if v is a desirable vertex for f, then there is a certain vertex $y \in N\{T_v\} \cap F_1$ such that $N(\{v, y\}) \cap F_3 = N(v) \cap F_3$ and either "$vy \in E(G)$ and y is not a desirable vertex" or "$T_v \cap T_y = T_v \backslash \{v\} = T_y \backslash \{y\}$ and y is a desirable vertex".*

Proof. Let $v \in F_1$ be a vertex of G with $|N(v) \cap F_3| = 2$. Let C be a triangle containing v. By the fact $v \in F_1$, v is either a (0, 3, 1)-vertex or a (1, 1, 2)-vertex.

If v is a (0, 3, 1)-vertex, then for each vertex $u \in C$, we have $f^*(u) \leqslant 2$. Consequently, $|N(v) \cap F_3| \leqslant 1$, a contradiction. Hence, v is a (1, 1, 2)-vertex.

If $C \cap V_{-1}$ is not empty, then, for each vertex $u \in C$, we have $f^*(u) \leqslant 2$, and so, $|N(v) \cap F_3| \leqslant 1$, a contradiction. Hence, $C \cap V_{-1} = \emptyset$. This implies that v is a single-triangle vertex. Now, suppose that $N(v) = \{u, x, y\}$ such that $f(y) = -1$. Then $C = T_v = \{v, u, x\}$ and y is a single-triangle vertex. By the fact that $f(y) = -1$, for each vertex $z \in T_y$, we have $f^*(z) \leqslant 2$. Hence $N(v) \cap F_3 = \{u, x\}$. We distinguish the following two subcases.

Case 1: $y \in F_1$. In this case, $|N(v) \cap F_3| = 2$ and $|N\{T_v\} \cap F_1| \geqslant 1$. So, v is a desirable vertex for f. Furthermore, $N(\{v, y\}) \cap F_3 = N(v) \cap F_3 = \{u, x\}$. Hence, (b) holds in this case. We can see that, in this case, y is not a desirable vertex for f.

Case 2: $y \in F_2$. In this case, since $y \in V_{-1} \cap F_2$, y is a (1, 0, 3)-vertex. It follows that $f(v) = 1$. Suppose, without loss of generality, that $f(u) = 1$ and $f(x) = 0$.

Case 2.1: There is a vertex $w \in N(\{u, x\}) \backslash \{v\}$ such that $w \in F_1$. If $wu, wx \in E(G)$, let z be the unique vertex in $N(w) \backslash \{u, x\}$. Since $f^*(w) = 1$, $f(u) = 1$, $f(x) = 0$ and $f^*(u) = f^*(x) = 3$, we have $f(w) = 1$ and $f(z) = -1$. This implies that $f^*(z) \leqslant 2$, and so, $N(\{v, w\}) \cap F_3 = N(v) \cap F_3 = \{u, x\}$. Then v is a desirable vertex for f and (b) holds in this subcase. It should be pointed out that, in this subcase, w is also a desirable vertex for f with $T_w \cap T_v = \{u, x\} \subseteq F_3$.

If exactly one of wu and wx is an edge of G, then w is a single-triangle vertex of G. Since $f^*(w) = 1$, $f(u) = 1$, $f(x) = 0$ and $f^*(u) = f^*(x) = 3$, we must have $f(w) = 1$ and $f(T_w) \leqslant 1$. This implies that $T_w \cap F_3 = \emptyset$, and so, $N(\{v, w\}) \cap F_3 = N(v) \cap F_3 = \{u, x\}$. Then v is a desirable vertex for f and (b) also holds in this subcase. We can see that, in this subcase, w is not a desirable vertex for f.

Case 2.2: $N\{T_v\} \cap F_1 = \emptyset$. In this subcase, v is an undesirable vertex. The facts $f(v) = 1$ and $f(x) = 0$ imply that (a) holds. □

Lemma 2.7. *There is a minimal minus dominating function g such that $g(V(G)) = f(V(G))$ and G has no undesirable vertices for g.*

Proof. If G has no undesirable vertex for f, we have nothing to do. Otherwise, let $B \subseteq V(G)$ be the set of undesirable vertices for f. By Lemma 2.6(a), for each undesirable vertex $v \in B$, we have $f(v) = 1$ and there is a vertex $v^* \in T_v$

such that $f(v^*) = 0$. Set

$$B^* = \{v^* : v \in B\}.$$

We define a new function $g : V(G) \to \{-1, 0, 1\}$ by the following way:

$$g(v) = \begin{cases} f(v) & \text{if } v \in V(G)\backslash(B \cup B^*), \\ 0 & \text{if } v \in B, \\ 1 & \text{if } v \in B^*. \end{cases}$$

Clearly, $g(V(G)) = f(V(G))$. By the facts that f is a minus dominating function and $N\{T_v\} \cap F_1 = \emptyset$ for every $v \in B$, we see that g is also a dominating function of G. Recall that f is a minimal minus dominating function, and so, by Lemma 2.1, for every $v \in V(G)$ with $f(v) \geqslant 0$, there is some $u \in N[v]$ such that $f^*(u) = 1$. Two favorable facts are

$$F_1 \subseteq \{v \in V(G) : g^*(v) = 1\}$$

and

$$\{v \in V(G) : f(v) \geqslant 0\} = \{v \in V(G) : g(v) \geqslant 0\}.$$

So, for every $v \in V(G)$ with $g(v) \geqslant 0$, there is some $u \in N[v]$ such that $g^*(u) = 1$. By, Lemma 2.1 again, g is a minimal minus dominating function. Our final observation is that G has no undesirable vertices for g. The result follows. $\square$

Based on the result of Lemma 2.7, we assume in the following that f is a minimal minus dominating function of G such that G has no undesirable vertices for f.

Lemma 2.8. $|F_1| \geqslant |F_3| + 2|F_4|$.

Proof. Let D be the set of desirable vertices of G for f. Set

$$\mathscr{D}^* = \{\{v, w\} \subseteq D : T_v \cap T_w = T_v\backslash\{v\} = T_w\backslash\{w\}\},$$

and

$$F_1(3) = \bigcup_{\{v,w\} \in \mathscr{D}^*} \{v, w\}.$$

By Lemma 2.6(b), for each vertex $v \in D\backslash F_1(3)$, there is a certain vertex $v' \in N\{T_v\} \cap (F_1\backslash D)$ with $vv' \in E(G)$ such that $N(\{v, v'\}) \cap F_3 = N(v) \cap F_3 = T_v\backslash\{v\}$. Set

$$F_1(2) = \{v, v' : v \in D\backslash F_1(3)\}.$$

It can be observed that $F_1(2)$, $F_1(3)$ and $F_1(4)$ are mutually disjoint. We further write

$$F_1(1) = F_1\backslash(F_1(2) \cup F_1(3) \cup F_1(4)).$$

By the definition of desirable vertices, Lemma 2.6(b) and Lemma 2.5, we have

$$\begin{aligned} &|F_1(1)| \geqslant |N(F_1(1)) \cap F_3|, \\ &|F_1(2)| = |N(F_1(2)) \cap F_3|, \\ &|F_1(3)| = |N(F_1(3)) \cap F_3|, \\ &|F_1(4)| \geqslant |N(F_1(4)) \cap F_3| + 2|F_4|. \end{aligned}$$

Consequently, we have

$$|F_1| \geqslant |N(F_1) \cap F_3| + 2|F_4|.$$

By Lemma 2.1, we have $|N(F_1) \cap F_3| = |F_3|$, and so the result follows. $\square$

Combining Lemmas 2.2 and 2.8, we deduce $f(V(G)) \leqslant \frac{1}{2}|V(G)|$. We conclude the main result of this paper as follows:

Theorem 2.9. *The upper minus domination number $\Gamma^-(G)$ of a claw-free cubic graph G is at most $\frac{1}{2}|V(G)|$.*

The bound established in Theorem 2.9 is sharp indeed. To see this, for an even positive integer k, we construct a claw-free cubic graph on $n = 3k$ vertices as follows:

$$V(G) = \{x_i : 1 \leqslant i \leqslant 2k\} \cup \{y_i : 1 \leqslant i \leqslant k\}$$

and

$$E(G) = E_1 \cup E_2 \cup E_3,$$

where

$$E_1 = \{x_1x_2, x_2x_3, \ldots, x_{2k-1}x_{2k}, x_{2k}x_1\},$$

$$E_2 = \{y_ix_{2i-1}, y_ix_{2i} : 1 \leqslant i \leqslant k\},$$

and

$$E_3 = \left\{ y_{2i-1}y_{2i} : 1 \leqslant i \leqslant \frac{k}{2} \right\}.$$

We define a minus dominating function f of G by setting

$$f(v) = \begin{cases} -1 & \text{if } v = y_{2i-1}, \quad 1 \leqslant i \leqslant \frac{k}{2}, \\ 0 & \text{if } v = y_{2i}, \quad 1 \leqslant i \leqslant \frac{k}{2}, \\ 1 & \text{if } v = x_i, \quad 1 \leqslant i \leqslant 2k. \end{cases}$$

It can be observed that, for each vertex $v \in V(G)$, we have $f^*(v) \geqslant 1$. So, f is indeed a minus dominating function of G. Furthermore, we can also observe that $f^*(y_i) = 1$ for each i with $1 \leqslant i \leqslant k$. By Lemma 2.1, f is a minimal minus dominating function. Since $f(V) = \frac{1}{2}|V(G)|$ is obvious, we deduce from Theorem 2.9 that $\Gamma^-(G) = \frac{1}{2}|V(G)|$.

Recall the following result implied in [6]: $\Gamma_s(G) \geqslant \gamma_s(G) \geqslant \frac{1}{2}|V(G)|$ for a cubic graph G. We have the following consequence.

Corollary 2.10. *If G is a claw-free cubic graph, then $\Gamma^-(G) \leqslant \Gamma_s(G)$.*

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